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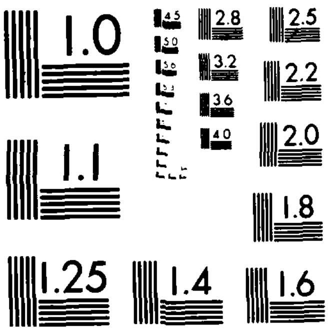

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WATER WAVES GENERATED BY A SLOWLY MOVING TWO-DIMENSIONAL BODY

Part II

Si-Xiong Chen
T. Francis Ogilvie

This research was carried out under the sponsorship of the Naval Sea Systems Command General Hydromechanics Research (GHR) Program under Contract N00014-81-K-0201

THE DEPARTMENT OF NAVAL ARCHITECTURE AND MARINE ENGINEERING

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A solution is obtained, valid asymptotically as speed approaches zero, for the waves generated behind a two-dimensional surface-piercing body moving ahead at constant speed U . The method of matched asymptotic expansions is used. There are two regions of interest: (i) In a thin surface layer behind the body but not contiguous to it, the generalized WKB method is used to determine the wave motion, except for a constant multiplicative factor. (ii) In a region behind the body and close to it, an integral equation is

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20. Abstract (Continued)

formulated and solved. This near-field solution can be determined completely from the condition that there must be a stagnation point at the intersection of the free surface and the body surface. Matching to the far-field solution then determines the unknown factor in the far-field solution. No radiation condition is available, since the thin surface layer of waves behind the body is completely isolated from any possible corresponding layer upstream. An asymptotic formula for wave resistance is found, in which the resistance is proportional to $C^{10}U^{48}$, where C is the body curvature at the intersection of the body and the undisturbed free surface. If $C = 0$, the power of U in the resistance formula is higher than 48; its value depends on what is the lowest non-zero derivative of body shape at the intersection. It is speculated that, for an analytically vertical body surface in some neighborhood of the intersection, the wave resistance is proportional to $\exp(-1/U)$ as U approaches zero.

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WATER WAVES GENERATED BY A

SLOWLY MOVING TWO-DIMENSIONAL BODY

Part II

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CONTENTS

PRINCIPAL NOMENCLATURE

[Note: Equation numbers are given below where it may help to identify the point of first introduction of a symbol.]

$A(x,y)$	Factor in $f(Z)$ (36)
A_0	Constant factor in $\tilde{\phi}_1$ (46)
$b(x)$	$iA(x,0)/\bar{\Phi}_x(x,\varepsilon\bar{H})$ (40)
$C(y)$	Body curvature (48d)
F	Froude number, $U/(gL)^{1/2}$
$\tilde{f}(Z)$	Complex potential in outer solution (31)
$G(X,Y;\xi,\eta)$	Green function used in inner region (63)
g	Gravitation constant
$H(x)$	$(1/\varepsilon)\times$(free-surface elevation)
$\bar{H}(x;\varepsilon)$	$(1/\varepsilon)\times$(free-surface elevation in "naive expansion") (9)
$\tilde{H}(x;\varepsilon)$	$(1/\varepsilon)\times$(free-surface elevation of wave motion) (9)
$\tilde{H}_j(X;\varepsilon)$	j-th term in expansion of $\tilde{H}$ in inner region (55)
$K(x)$	$d\theta(x)/dx$ (14)
L	Typical body dimension
n	Unit normal to body surface, directed into the body
U	Forward speed
u_0	Constant $= -C(0)\phi_{0_{xx}}(x_0,0)$ (48c)
$W(Z)$	Complex function defined in outer solution (33)
X	Θ/K (27)
x	Horizontal Cartesian coordinate
x_0	x-coordinate of downstream intersection of body and undisturbed free surface or, approximately, of downstream stagnation point
Y	y/ε in outer solution (13); $[y-\varepsilon\bar{H}(x)]/\varepsilon^5$ in inner solution (52)
Y	$Y-\bar{H}$ in outer solution (27)
y	Vertical Cartesian coordinate
Z	$Z+i\hat{Y}$ in outer solution (30)
ι	Exponent of ε in outer solution (10),(11)
ε	Small parameter of the problem $= F^2 = U^2/gL$
$\eta_j(x;\varepsilon)$	Term in expansion of $\tilde{H}(x;\varepsilon)$ (11)

$\Theta(x)$	Rapidly varying phase function in outer solution (12)
$\theta(x)$	$\varepsilon\Theta(x)$ (12)
$\mu(Y)$	Source density on body surface (65)
ν	Nondimensional wave number = $1/u_0^2$
$\Phi(x,y)$	Velocity potential of complete problem
$\bar{\Phi}(x,y;\varepsilon)$	Velocity potential in "naive expansion" (8)
$\tilde{\Phi}(x,y;\varepsilon)$	Velocity potential for wave motion (8)
$\tilde{\Phi}_j(X,Y;\varepsilon)$	Term in expansion of $\tilde{\Phi}$ in inner region (54)
$\phi_j(x,y;\Theta,Y)$	Term in expansion of $\tilde{\Phi}(x,y;\varepsilon)$ in outer region (10)
$\Phi_0(x,y)$	Velocity potential in double-body problem (48e)

I. INTRODUCTION

There have been two kinds of methods published for treating the ship/wave problem for a ship moving at very low speed:

(1) First Ogilvie (1968) used an order-of-magnitude argument to obtain a linear free-surface condition that would lead to the prediction of a plausible wave motion at very low speeds. There were two essential points: (a) the waves should have very short wavelength, and (b) the waves should propagate on the nonlinear streaming flow around the corresponding double body. In the linear wave problem, the apparent cause of wave generation is an effective pressure distribution on the free surface, which arises mathematically because the double-body flow does not really satisfy the precise free-surface conditions. Ogilvie treated only the case of a fully submerged two-dimensional body. Later, Baba and Takekuma (1975) extended this concept to solve the problem of a three-dimensional surface-piercing body, e.g., a ship. They went so far as to derive a wave-resistance formula based on this approach. Maruo and Fukazawa (1979) extended this approach further, using a coordinate transformation to simplify the analysis.

(2) Keller (1974) developed the first ray theory for the low-speed problem. Inui and Kajitani (1977) used a procedure based on a method of Ursell's (1960), which is essentially a ray method. Later, Keller (1979) further developed his ray method with systematic asymptotic expansions, and he applied his theory to a thin ship and a special class of "streamlined" ships. In Keller's ray theory, the waves are apparently generated only at the stagnation points on the body; the amplitude and phase of the waves are then modified gradually by the nonuniform flow around the double body.

There are difficulties in both methods. In the first, the linear free-surface condition can be written in such a way that the terms on, say, the left-hand side are rapidly varying wavelike quantities, while those on the right-hand side are slowly varying in space. The latter are completely known; they represent the fictitious pressure field imposed on the free surface. The situation can be compared to the much simpler problem cited by Keller (1979) (see his Appendix A):

$$u''(x) + k^2 u(x) = g(x) ,$$

where $g(x)$ is analogous to the fictitious pressure distribution. If k is very large, wave solutions of this differential equation represent very short waves. The general solution of the above equation can, of course, be written out explicitly, and it can then be expanded asymptotically for $k \to \infty$. If the domain of x is $-\infty < x < +\infty$, the only part of the asymptotic expansion that represents waves comes from the homogeneous problem, and its amplitude and phase can be determined generally only if somehow they are known at some point, possibly at infinity. In addition, there is a particular solution, which can be represented asymptotically as a series in inverse powers of k^2 ; it represents a slowly varying solution if $g(x)$ is slowly varying (as assumed). If the domain is restricted to, say, $0 \leq x < \infty$, the wave part of the solution depends entirely on the values of $g(x)$ and its derivatives at $x = 0$ and on the two boundary conditions imposed on $u(x)$. So we can say that the generation of waves in such a case is unaffected by the function $g(x)$ except in a neighborhood of $x = 0$. This raises a doubt about the fundamental supposition of the first method, namely, that the waves are generated (mathematically speaking) by the fictitious pressure distribution on the free surface. All that matters is the behavior of $g(x)$ near $x = 0$.

Dagan (1972) pointed out that the base flow on which the waves propagate should not be just the double-body flow, as assumed by Ogilvie and others, but at least two terms in the "naive expansion," which is the expansion that is obtained if the problem is expanded formally and strictly in terms of power series in the Froude number. Keller (1979) arrived at the same conclusion and noted further that then the fictitious pressure distribution vanishes from the wave problem. That is, the free-surface condition becomes homogeneous.

In fact, in Keller's (1979) ray theory, the wave part of the velocity potential function satisfies the Laplace equation, which is homogeneous, as well as homogeneous free-surface and body boundary conditions. Thus any solution that is found can be multiplied by an arbitrary constant. Keller introduced a so-called "excitation coefficient" for certain simple special cases. Still, his method fails near the stagnation point of the double-body flow: From the dispersion relation, the wave number becomes inifinte there,

and the amplitude of the waves becomes infinite too. Since, as Keller himself pointed out, the generation of waves in the short-wave problem depends essentially on conditions at the boundary point (consider again the simple differential-equation problem cited above), the failure of the ray-theory assumptions near a stagnation point seems to be crucial.

We present here the second part of a study to resolve these questions. The first part is reported by Ogilvie and Chen (1982); it will be referred to subsequently simply as "I". They derived a nonhomogeneous body boundary condition for the wave part of the potential function. Their free-surface condition is homogeneous, as required from the work of Keller. The nonhomogeneity of the body boundary condition is, as we shall show, adequate for determining the solution without any arbitrary additive or multiplicative quantities.

We use the method of matched asymptotic expansions to solve the low-speed problem in two dimensions. Our outer region is a thin layer near the free surface, far behind the body (in terms of wavelength). We use the generalized WKB method to determine the nature of the wave motion in this region; this method is very similar to a ray method. Then we formulate a near-field problem, applicable to a very small region near the stagnation point. Considerations of the flow properties near a stagnation point are found to be sufficient for determining the near-field solution completely. Then matching to the outer solution permits the latter to be determined as well.

The final formula for the wave resistance, Equation (89), is surprising: Wave resistance is proportional to U^{48}, where U is the forward speed. We believe that this is the correct asymptotic relationship, although it is not likely to be useful to a naval architect. It clearly has no range of validity in U in which its predictions overlap those from conventional wave-resistance analyses. What is still needed is a small-U solution that gives this formula as $U \to 0$ and gives the results of conventional linear thoery as $U \to \infty$.

There is an even stronger contrast between the low-speed theory and conventional linear theory in the case of a submerged body. (The low-speed theory predicts that wave resistance is proportional to $e^{-1/U}$ as $U \to 0$.)

Tulin (1982) has produced an exact theory that bridges the two. It will be much more difficult to find a comparably general theory for the case of a surface-piercing body.

II. OUTER SOLUTION BY THE *WKB* METHOD

We want to find a velocity potential $LU\Phi(x,y)$ for the streaming flow past a two-dimensional body that intersects the free surface. The stream has speed U in the positive-x direction. The undisturbed free surface lies on the x axis, with the y axis directed upwards. We take the origin of coordinates inside the body in such a way that the intersections of the body and the undisturbed free surface are located at $(-x_0,0)$ and $(x_0,0)$. The latter is the one of primary interest in this paper, since it lies on the downstream side of the body. All length dimensions have been normalized with respect to L, any convenient characteristic length of the body. The small parameter of the problem is taken as

$$\varepsilon = F^2 = U^2/gL , \qquad (1)$$

where g is the gravitational constant and F is a Froude number. The shape of the free surface is given by a relationship $y = \varepsilon H(x)$, where $H(x)$ is to be determined as part of the solution of the problem.

The statement of the problem is as follows:

[L] $$\Phi_{xx} + \Phi_{yy} = 0 \quad \text{in the fluid domain;} \qquad (2)$$

[H] $$H(x) = \frac{1}{2}\left\{1 - \Phi_x^2 - \Phi_y^2\right\}\Big|_{y=\varepsilon H(x)} ; \qquad (3)$$

[K] $$\Phi_y = \varepsilon H_x \Phi_x \quad \text{on} \quad y = \varepsilon H(x) ; \qquad (4)$$

[F] $$\Phi_y + \varepsilon\left\{\Phi_x^2\Phi_{xx} + 2\Phi_x\Phi_y\Phi_{xy} + \Phi_y^2\Phi_{yy}\right\} = 0 \quad \text{on} \quad y = \varepsilon H(x) ; \qquad (5)$$

[B] $$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on the body;} \qquad (6)$$

[R] $$|\Phi - x| \to 0 \quad \text{as} \quad x \to -\infty \quad \text{and/or} \quad y \to -\infty . \qquad (7)$$

These are the same as in (I), although the [K] condition was not explicitly used there. Either the [F] condition or the [K] condition is redundant.

We assume that the solution can be divided into two parts:

$$\Phi(x,y) = \bar{\Phi}(x,y;\varepsilon) + \tilde{\Phi}(x,y;\varepsilon) \quad ; \tag{8}$$

$$H(x) = \bar{H}(x;\varepsilon) + \tilde{H}(x;\varepsilon) \quad . \tag{9}$$

The first part, represented by $\bar{\Phi}(x,y;\varepsilon)$ and $\bar{H}(x;\varepsilon)$, is the so-called *naive expansion* (see (I)); it is the formal solution that is obtained by simply substituting a power series in ε into the conditions (2) - (7). It does not represent a wavelike motion, although there is a corresponding free-surface deformation near the body, which vanishes downstream as well as upstream. The other terms in (8) and (9) represent true wave motions (suggested by the notation $\tilde{}$). We further assume that the wave part of the solution can be expanded as follows:

$$\tilde{\Phi}(x,y;\varepsilon) \sim \varepsilon^{\alpha+1}\tilde{\phi}_1(x,y;\Theta,Y) + \varepsilon^{\alpha+2}\tilde{\phi}_2(x,y;\Theta,Y) + \dots ; \tag{10}$$

$$\tilde{H}(x;\varepsilon) \sim \varepsilon^{\alpha}\tilde{\eta}_1(x;\Theta) + \varepsilon^{\alpha+1}\tilde{\eta}_2(x;\Theta) + \dots . \tag{11}$$

Three new quantities have been introduced:

(i) α is a real number greater than unity. In (I), it was taken as 1 . Actually, that is simply the smallest possible value of α that leads to a linear problem for the wave motion. Now we must use the nonhomogeneous boundary condition developed in (I) to determine the correct value of α . Note that the difference in the powers of ε between (10) and (11) results from the [H] condition, (3) .

(ii) $\Theta(x)$ is a rapidly varying phase function, which we shall also write in the form

$$\Theta(x) = \theta(x)/\varepsilon \quad . \tag{12}$$

We shall assume that $\theta(x)$ is slowly varying in the sense that its derivative is of the same order of magnitude as θ itself.

(iii) Y is a stretched coordinate:

$$Y = y/\varepsilon \quad . \tag{13}$$

Effectively, we treat this as a multiple-scale problem, x and y being used to describe changes that occur on a scale comparable to the dimensions of the body, Θ and Y being used to describe the details on the scale of the wavelength. We imply here that the generated waves have wavelength that is $O(\varepsilon)$. This is valid in the *outer* region, which means a region many wavelengths away from the downstream stagnation point. The implication is not valid very near to the stagnation point. The latter case is discussed in the next section.

The expansions (10) and (11) are called *generalized WKB expansions*. The functions $\tilde{\phi}_i$ and $\tilde{\eta}_i$ all represent wave motions that are superposed on a nonuniform, nonwavelike base flow given by $\bar{\Phi}$ and $\bar{H}$.

Let us define

$$K(x) = \frac{d\theta(x)}{dx} . \tag{14}$$

We note the following formulas for differentiations:

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\bar{\Phi}}{\partial x} + \varepsilon^{\alpha+1}\{\tilde{\phi}_{1_x} + [K/\varepsilon]\tilde{\phi}_{1_\Theta}\} + \varepsilon^{\alpha+2}\{\tilde{\phi}_{2_x} + [K/\varepsilon]\tilde{\phi}_{2_\Theta}\} + \ldots ; \tag{15}$$

$$\frac{\partial\Phi}{\partial y} = \frac{\partial\bar{\Phi}}{\partial y} + \varepsilon^{\alpha+1}\{\tilde{\phi}_{1_y} + [1/\varepsilon]\tilde{\phi}_{1_Y}\} + \varepsilon^{\alpha+2}\{\tilde{\phi}_{2_y} + [1/\varepsilon]\tilde{\phi}_{2_Y}\} + \ldots ; \tag{16}$$

$$\frac{dH}{dx} = \frac{d\bar{H}}{dx} + \varepsilon^{\alpha}\{\tilde{\eta}_{1_x} + [K/\varepsilon]\tilde{\eta}_{1_\Theta}\} + \varepsilon^{\alpha+1}\{\tilde{\eta}_{2_x} + [K/\varepsilon]\tilde{\eta}_{2_\Theta}\} + \ldots . \tag{17}$$

Now we substitute (8)-(9) and (10)-(11) into the conditions of the problem, starting with the Laplace equation (2). Then we rearrange terms according to powers of ε and set the coefficient of each power separately equal to zero. From the coefficient of $\varepsilon^{\alpha-1}$, we obtain:

$$[L] \qquad K^2\tilde{\phi}_{1\Theta\Theta} + \tilde{\phi}_{1YY} = 0 . \tag{18}$$

Similarly, from the coefficient of ε^{α} , we obtain:

$$[L] \qquad K^2\tilde{\phi}_{2\Theta\Theta} + \tilde{\phi}_{2YY} = -\left\{(K\tilde{\phi}_{1\Theta})_x + K\tilde{\phi}_{1\Theta x} + 2\tilde{\phi}_{1yY}\right\} . \tag{19}$$

The letter subscripts indicate partial or total derivatives, as appropriate. (For example, $K_x = dK/dx$, whereas $\phi_x = \partial\phi/\partial x$.)

Next we substitute into the [H] condition, (3). Initially, all functions of y and of Y must be evaluated on $\varepsilon H(x)$ and on $H(x)$, respectively. Then we expand these functions in Taylor series as follows: $\bar{\Phi}$ (and its derivatives) is expanded with respect to $y = \varepsilon\bar{H}$, whereas $\tilde{\phi}_i$ is expanded with respect to $y = 0$ and $Y = \bar{H}$. (The last is permissible if, as assumed, $\alpha > 1$.) We note that, from the definition of the naive expansion,

$$\bar{\Phi}_y(x,\varepsilon\bar{H}) = \varepsilon\bar{H}_x\bar{\Phi}_x(x,\varepsilon\bar{H}) . \tag{20}$$

From the coefficient of ε^{α} in (3), we obtain:

$$[H] \qquad \tilde{\eta}_1(x,\Theta) = -K(x)\bar{\Phi}_x(x,\varepsilon\bar{H})\tilde{\phi}_{1\Theta}(x,0;\Theta,\bar{H}) \; . \qquad (21)$$

Similarly, from the coefficient of $\varepsilon^{\alpha+1}$,

$$[H] \qquad \tilde{\eta}_2 = -\bar{\Phi}_x\{K\tilde{\phi}_{2\Theta} + \tilde{\phi}_{1x} + K\bar{H}\tilde{\phi}_{1\Theta y} + \bar{H}_x\tilde{\phi}_{1Y}\} \quad \text{on} \quad y = 0 \; , \quad Y = \bar{H} \; . \qquad (22)$$

Here and throughout this section, the notation " $y = 0$ ", as in (22), refers only to $\tilde{\phi}_i$. As already mentioned, and as indicated explicitly in (21), we evaluate $\bar{\Phi}$ on $y = \varepsilon\bar{H}$.

Following the same procedure with the [K] condition, (4), we obtain:

$$[K] \qquad \tilde{\phi}_{1Y} = K\bar{\Phi}_x\tilde{\eta}_{1\Theta} \qquad \text{on} \quad y = 0 \; , \quad Y = \bar{H} \; ; \qquad (23)$$

$$[K] \quad \tilde{\phi}_{1y} + \tilde{\phi}_{2Y} + \bar{\Phi}_{yy}\tilde{\eta}_1 = \bar{\Phi}_x\tilde{\eta}_{1x} + K\bar{H}_x\tilde{\phi}_{1\Theta} + K\bar{\Phi}_x\tilde{\eta}_{2\Theta} - \tilde{\phi}_{1yY}\bar{H} \quad \text{on} \quad \begin{cases} y = 0 \; , \\ Y = \bar{H} \; . \end{cases} \qquad (24)$$

We can eliminate $\tilde{\eta}_1$ between (21) and (23) (or, alternatively, start with the [F] condition, (5)). Then the $\tilde{\phi}_1$ problem is given by the following:

$$[L] \qquad K^2\tilde{\phi}_{1\Theta\Theta} + \tilde{\phi}_{1YY} = 0 \qquad \text{in} \quad y \le 0 \; , \quad Y \le \bar{H} \; ; \qquad (18)$$

$$[F] \qquad \tilde{\phi}_{1Y} + K^2\bar{\Phi}_x^2\tilde{\phi}_{1\Theta\Theta} = 0 \qquad \text{on} \quad y = 0 \; , \quad Y = \bar{H} \; . \qquad (25)$$

Then we eliminate $\tilde{\eta}_2$ between (22) and (24), which, together with (19), gives the $\tilde{\phi}_2$ problem:

$$[L] \quad K^2\tilde{\phi}_{2\Theta\Theta} + \tilde{\phi}_{2YY} = -\{(K\tilde{\phi}_{1\Theta})_x + K\tilde{\phi}_{1\Theta x} + 2\tilde{\phi}_{1yY}\} \quad \text{in} \quad y \le 0 \; , \quad Y \le \bar{H} \; ; \qquad (19)$$

$$[F] \quad \tilde{\phi}_{2Y} + K^2\bar{\Phi}_x^2\tilde{\phi}_{2\Theta\Theta} = -\tilde{\phi}_{1y} + \bar{\Phi}_x\tilde{\eta}_{1x} + K\bar{H}_x\tilde{\phi}_{1\Theta} - \bar{H}\tilde{\phi}_{1yY} - K\bar{\Phi}_x^2\tilde{\phi}_{1x\Theta}$$
$$- K^2\bar{\Phi}_x^2\bar{H}\tilde{\phi}_{1\Theta\Theta y} - K\bar{\Phi}_x^2\bar{H}_x\tilde{\phi}_{1Y\Theta} - \bar{\Phi}_{yy}\tilde{\eta}_1 \qquad \text{on} \quad \begin{cases} y = 0 \; , \\ Y = \bar{H} \; . \end{cases} \qquad (26)$$

Now let us solve the problem stated in (18) and (25). The independent variables are Θ and Y ; we can consider x and y as if they were parameters. By a minor change of variables, we transform the first equation into the Laplace equation. Let

$$X = \Theta/K \; , \quad \hat{Y} = Y - \bar{H} \; . \qquad (27)$$

Then (18) and (25) become:

$$\tilde{\phi}_{1XX} + \tilde{\phi}_{1\hat{Y}\hat{Y}} = 0 \quad \text{in} \quad y \le 0 \;, \quad \hat{Y} \le 0 \;; \tag{28}$$

$$\tilde{\phi}_{1\hat{Y}} + \bar{\Phi}_x^2 \tilde{\phi}_{1XX} = 0 \quad \text{on} \quad y = 0 \;, \quad \hat{Y} = 0 \;. \tag{29}$$

The slight change from Y to $\hat{Y}$ in (25) has enabled us to formulate a problem with a free-surface condition given on the X axis.

We introduce complex variables Z and $\tilde{f}(Z)$:

$$Z = X + i\hat{Y} \;, \tag{30}$$

$$\tilde{\phi}_1 = Re\{\tilde{f}(Z)\} \;. \tag{31}$$

(Of course, $\tilde{f}$ also depends on x and y , but we continue to treat the latter as parameters.) The Laplace equation, (28), is automatically satisfied when $\tilde{\phi}_1$ is defined as in (31). The free-surface condition, (29), can now be rewritten:

$$Re\left\{\bar{\Phi}_x^2(x,\varepsilon\bar{H})\tilde{f}'' + i\tilde{f}'\right\} = 0 \quad \text{on} \quad y=0 \;, \quad \hat{Y} = 0 \;. \tag{32}$$

Since we consider x (and thus $\bar{H}(x)$) as being fixed in (32), this condition is valid for all X . Thus we can define a new function

$$W(Z) = \bar{\Phi}_x^2(x,\varepsilon\bar{H})\tilde{f}''(Z) + i\tilde{f}'(Z) \;, \tag{33}$$

which can be continued analytically into the upper half plane as follows:

$$W(\bar{Z}) = -\overline{W(Z)} \;, \tag{34}$$

where the bars here denote complex conjugates. Since $W(Z)$ is analytic in the entire lower half of the Z plane, we now conclude that it is analytic in the entire plane. Thus it must be a constant. From (29), its real part vanishes, and, without loss of generality, we set its imaginary part equal to zero also. Then

$$\bar{\Phi}_x^2\tilde{f}''(Z) + i\tilde{f}'(Z) = 0 \quad \text{in the} \quad Z \quad \text{plane.} \tag{35}$$

This is an ordinary differential equation for $\tilde{f}(Z)$, which is easily solved:

$$\tilde{f}(Z) = A(x,y)\exp\left\{-iZ/\bar{\Phi}_x^2\right\} , \qquad (36)$$

where $A(x,y)$ is an arbitrary constant with respect to X and $\hat{Y}$. In general, an additive constant, say $C(x,y)$, can be added to (36), but it contributes nothing to the wave solution, and so we set it equal to zero.

We require that $\tilde{f}$ be a periodic function of Θ . We have not yet specified $\Theta = \varepsilon\Theta(x)$, and so, without loss of generality, we can require that the period be 2π . Substituting $Z = \Theta/K + i\hat{Y}$ into (36) and taking into account that $K = \theta'(x) = \varepsilon\Theta'(x)$, this requirement is equivalent to the following:

$$-\frac{\Theta}{\varepsilon\bar{\Phi}_x^2\Theta'} + 2\pi = -\frac{\Theta+2\pi}{\varepsilon\bar{\Phi}_x^2(\Theta+2\pi)'} .$$

It then follows that

$$\Theta(x) = -\frac{1}{\varepsilon}\int_{x_0}^{x} d\xi/\bar{\Phi}_x^2(\xi,\varepsilon\bar{H}(\xi)) , \qquad (37)$$

where once again, without loss of generality, we have set a constant of integration equal to zero. Substituting into (36) and then using (31), we have the solution:

$$\tilde{\phi}_1(x,y;\Theta,Y) = Re\{\tilde{f}(Z)\} = Re\left\{A(x,y)\exp\left((Y-\bar{H})/\bar{\Phi}_x^2\right)\exp\left(i\Theta(x)\right)\right\} . \qquad (38)$$

From (21) we obtain the wave-elevation function:

$$\tilde{\eta}_1(x;\Theta) = Re\left\{b(x)\exp\left(i\Theta(x)\right)\right\} , \qquad (39)$$

where

$$b(x) = iA(x,0)/\bar{\Phi}_x(x,\varepsilon\bar{H}) . \qquad (40)$$

From now on, for convenience, we drop the notation Re , but we imply that it should be included in expressions like (38) and (39).

The situation represented by the solution $\tilde{\phi}_1$ and $\tilde{\eta}_1$ above is familiar in applications of the WKB method: We now know the basic form of the wave solution, but we do not know its amplitude anywhere. We know the phase function, $\Theta(x)$, but the [complex] amplitude $A(x,0)$ is completely unknown.

In order to obtain more information about $A(x,0)$, we must consider the second-order problem, given by (19) and (26). The form of the left-hand sides

of (19) and (26) is identical to that of (18) and (25), which gave the first-order problem. So we can expect the second-order solution to represent waves like those described in (38) and (39). That is, $\tilde{\phi}_2$ will contain a part that satisfies the homogeneous counterpart of (26), and this part will depend on the same phase function as that in $\tilde{\phi}_1$. In addition, we note that the right-hand sides of (19) and (26) are linear in $\tilde{\phi}_1$ and $\tilde{\eta}_1$, and so the solution of the nonhomogeneous problem will also involve the same phase function. So we now write:

$$\tilde{\phi}_2 \;=\; A_2(x,y)\exp\{(y-\bar{H})/\bar{\Phi}_x^2\}\exp\{i\Theta(x)\} \quad , \tag{41}$$

where $\Theta(x)$ is still given by (37) and $A_2(x,y)$ is an unknown function.

When (41) is substituted into (19) and (26), it is evident that the equations can be satisfied only if the right-hand sides of those equations are separately equal to zero. This provides the further conditions needed for determining the first-order solution. Substituting (38) and (39) into the right-hand sides of (19) and (26), setting them equal to zero, and letting $y = \bar{H}$, we find that:

$$iAK_x + 2iKA_x - 2ikA\bar{H}_x/\bar{\Phi}_x^2 + 2A_y/\bar{\Phi}_x^2 \;=\; 0 \;, \tag{42}$$

$$\bar{\Phi}_x b_x - \bar{\Phi}_{yy} b - 2i\bar{H}_x A/\bar{\Phi}_x^2 - i\bar{\Phi}_{xx}A/\bar{\Phi}_x \;=\; 0 \;, \tag{43}$$

where we have used the fact that $\bar{H}_x = -\bar{\Phi}_x\bar{\Phi}_{xx} + O(\varepsilon)$, which follows readily from (3) and (4). We use this relationship again in (42), noting also that $K = -1/\bar{\Phi}_x^2$, to obtain the following:

$$A_y(x,0) \;=\; iA_x(x,0) \;. \tag{44}$$

Then we use this result and the definition (40) in (43) to obtain:

$$A_x + A\bar{\Phi}_{xx}/\bar{\Phi}_x \;=\; 0 \;. \tag{45}$$

This differential equation is easily solved: $A(x,0) = A_0/\bar{\Phi}_x$, where A_0 is strictly a constant. Thus we have found the form of $A(x,0)$, and so we have for $\tilde{\phi}_1$:

$$\tilde{\phi}_1(x,0;\Theta,\bar{H}) \;=\; \{A_0/\bar{\Phi}_x(x,\varepsilon\bar{H})\}\, e^{i\Theta(x)} \;. \tag{46}$$

The constant A_0 can only be found from matching this solution with a near-field (inner) solution. This situation arises because our fundamental equation is an elliptic equation (the Laplace equation), and the typical WKB wave approximation is not valid in a region near the body, where elliptic behavior dominates the wave behavior.

III. INNER SOLUTION BY THE SOURCE-DISTRIBUTION METHOD

The solution obtained in the preceding section represents a wave motion with very short waves. To be precise, from (37) and (38), it is evident that the local wave number is

$$\Theta'(x) = -1/\varepsilon\bar{\Phi}_x^2(x,\varepsilon\bar{H}(x)) . \tag{47}$$

Since, in general, $\bar{\Phi}_x = O(1)$ as $\varepsilon \to 0$, we have

$$\Theta'(x) = O(1/\varepsilon) . \tag{47'}$$

The assumption in (12) was really an anticipation of this condition.

If, however, $\bar{\Phi}_x$ vanishes at some point, the wave number in (47) is undefined at that point. Our conclusions must be reconsidered in a neighborhood of that point.

We expect that there will be a stagnation point on the downstream side of the body, presumably at the intersection of the body and the free surface. Such a stagnation point will be located at $y = \varepsilon/2$, as shown by (3). We set $x = x_0$ at this point.*

At the stagnation point, we have $\phi_x = \phi_y = 0$. If, separately, we require that $\bar{\Phi}_x = \bar{\Phi}_y = 0$ at the stagnation point, we create precisely the condition mentioned above: The wave number is undefined. So we consider more carefully the behavior of $\bar{\Phi}$ near $(x_0,\varepsilon/2)$. In Appendix A we show that

$$\bar{\Phi}_x(x_0,\varepsilon/2) = \varepsilon^2 u_0 + o(\varepsilon^2) , \tag{48a}$$

$$\bar{\Phi}_y(x_0,\varepsilon/2) = O(\varepsilon^5) , \tag{48b}$$

where

$$u_0 = -C(0)\phi_{0xx}(x_0,0) , \tag{48c}$$

$$C(0) = \text{body curvature at } y = 0 , \tag{48d}$$

$$\phi_0(x,y) = \text{first term in an } \varepsilon \text{ expansion of } \bar{\Phi}(x,y).^\dagger \tag{48e}$$

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*As in (I), we shall also take $x = x_0$ at the downstream intersection of the body and the x axis. Since the body is assumed to be smooth and to have a vertical tangent at $y = 0$, this practice should not cause significant error.

†$\phi_0(x,y)$ is the solution of the "rigid-wall" or "double-body" problem, in which the free surface is replaced by a rigid wall at $y = 0$. See (I) for details.

From these relationships, we now show that the wave number is $O(1/\varepsilon^5)$ in a small region near $(x_0, \varepsilon/2)$.

We substitute (8) into the free-surface condition (5), eliminate the terms that involve only the naive expansion, and then determine the leading-order wavelike terms. In view of the estimates (48a) and (48b), we can, to leading order, drop all terms except the following:

$$\tilde{\Phi}_y + \varepsilon\bar{\Phi}_x^2\tilde{\Phi}_{xx} \sim 0 \quad .$$

This is directly comparable to the problem statement in (I), and it is also equivalent to (29) above. However, it is now being used in a small region near the stagnation point, where, from (48a), $\bar{\Phi}_x = O(\varepsilon^2)$. (In other words, we now use this condition *everywhere* for obtaining the lowest-order term in the wavelike solution.) Thus we have

$$\tilde{\Phi}_y + \tilde{\Phi}_{xx} \cdot O(\varepsilon^5) \sim 0 \; .$$

These two terms must be of the same order of magnitude, for otherwise $\tilde{\Phi}$ would not represent a wave motion. Furthermore, since the potential satisfies the Laplace equation, $\partial/\partial x$ and $\partial/\partial y$ should have similar order-of-magnitude effects on $\tilde{\Phi}$. These requirements can be satisfied only if

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y} = O(\varepsilon^{-5}) \tag{49}$$

when acting on $\tilde{\Phi}$, which is equivalent to the statement that wave number is $O(\varepsilon^{-5})$. This is valid only in a region near the stagnation point.

From (48a) we can also determine the order of magnitude of $\tilde{\Phi}$ in the neighborhood of the stagnation point. The complete potential, $\bar{\Phi} + \tilde{\Phi}$, must give no normal velocity component on the body. However, from (48a), $\bar{\Phi}_x = O(\varepsilon^2)$ at the stagnation point, and so $\partial\bar{\Phi}/\partial n = O(\varepsilon^2)$ there. In order to cancel this, we must have $\partial\tilde{\Phi}/\partial n = O(\varepsilon^2)$ too at the stagnation point. But, in view of (49), this is possible only if

$$\tilde{\Phi} = O(\varepsilon^7) \tag{50}$$

near the stagnation point. (Note that this still does not give us α in (10) and (11). Those expressions are not valid near the stagnation point.)

The next problem is to formulate and solve a precise near-field problem for matching with the outer solution from Section II.

We define the near field as a region in which

$$\left.\begin{aligned} x - x_0 &= O(\varepsilon^5) \, , \\ y - \varepsilon\bar{H}(x) &= O(\varepsilon^5) \, . \end{aligned}\right\} \qquad (51)$$

It is sometimes convenient to define new near-field variables:

$$\left.\begin{aligned} X &= (x - x_o)/\varepsilon^5 \, , \\ Y &= [y - \varepsilon\bar{H}(x)]/\varepsilon^5 \end{aligned}\right\} \qquad (52)$$

We note the following rules for differentiation:

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \varepsilon^{-5}\frac{\partial}{\partial X} - \varepsilon^{-4}\bar{H}'(x)\frac{\partial}{\partial Y} \, , \\ \frac{\partial}{\partial y} &= \varepsilon^{-5}\frac{\partial}{\partial Y} \, . \end{aligned}\right\} \qquad (53)$$

Since we shall solve the near-field problem just to one order of magnitude, we shall have to use only the first term on the right-hand side of the first formula of (53).

In the near field, we expand the potential:

$$\Phi(x,y) = \bar{\Phi}(x,y;\varepsilon) + \varepsilon^7\tilde{\Phi}_1(X,Y;\varepsilon) + \ldots \, . \qquad (54)$$

From this expansion and (3), we find that we can also write:

$$H(x) = \bar{H}(x;\varepsilon) + \varepsilon^4\tilde{H}_1(X;\varepsilon) + \ldots \, . \qquad (55)$$

The free-surface condition, (5), can now be expanded as follows:

$$\begin{aligned} 0 &= \bar{\Phi}_y + \varepsilon^7\tilde{\Phi}_{1_y} + \ldots + \varepsilon\left\{[\bar{\Phi}_x + \varepsilon^7\tilde{\Phi}_{1_x} + \ldots]^2[\bar{\Phi}_{xx} + \varepsilon^7\tilde{\Phi}_{1_{xx}} + \ldots] + \ldots\right\} \\ &= \bar{\Phi}_y + \varepsilon\bar{\Phi}_x^2\bar{\Phi}_{xx} + \ldots + \varepsilon^2\tilde{\Phi}_{1Y} + \ldots + \varepsilon\left\{\bar{\Phi}_x^2\cdot\varepsilon^{-3}\tilde{\Phi}_{1XX} + \ldots\right\} \, , \end{aligned} \qquad (56)$$

to be satisfied on $y = \varepsilon H = \varepsilon\bar{H} + \varepsilon^5\tilde{H}_1 + \ldots$, which is equivalent to $Y = \tilde{H}_1 + \ldots$. In the near field, $\bar{\Phi}$ and its derivatives can be evaluated at $x = x_0$, $y = \varepsilon/2$ with negligible (higher-order) error. Thus, for example,

$$\bar{\Phi}_x(x,\varepsilon H) = \bar{\Phi}_x(x_0,\varepsilon/2) + (x-x_0)\bar{\Phi}_{xx}(x_0,\varepsilon/2) + \varepsilon(H-\tfrac{1}{2})\bar{\Phi}_{yy}(x_0,\varepsilon/2) + \ldots$$
$$= \bar{\Phi}_x(x_0,\varepsilon/2) + \varepsilon^5 X\bar{\Phi}_{xx} + [\varepsilon^5\tilde{H}_1 + \varepsilon^6 X\bar{H}'(x_0) + \ldots]\bar{\Phi}_{yy} + \ldots$$
$$\sim \bar{\Phi}_x(x_0,\varepsilon/2) \sim \varepsilon^2 u_0 \qquad (57)$$

(See (48a)). The free-surface condition becomes, to leading order,

$$\tilde{\Phi}_{1Y} + u_0^2\tilde{\Phi}_{1XX} = 0 \qquad \text{on} \quad Y = \tilde{H}_1 + \ldots . \qquad (58)$$

The body boundary condition is as follows:

$$\frac{\partial\Phi}{\partial n} = \frac{\partial\bar{\Phi}}{\partial n} + \varepsilon^7\frac{\partial\tilde{\Phi}_1}{\partial n} + \ldots = 0 .$$

From the naive expansion, worked out in (I), we have

$$\frac{\partial\bar{\Phi}}{\partial n} = 0 \qquad \text{on body,} \quad y < 0 ,$$

$$\frac{\partial\bar{\Phi}}{\partial x} = -2\varepsilon y C(0)\phi_{0xx}(x_0,0) = 2\varepsilon y u_0 \qquad \text{on body,} \quad 0 < y < \varepsilon/2 .$$

Thus we require that

$$\varepsilon^7\frac{\partial\tilde{\Phi}_1}{\partial n} = 0 \qquad \text{on body,} \quad y < 0 , \qquad (59a)$$

$$\varepsilon^7\frac{\partial\tilde{\Phi}_1}{\partial x} = -2\varepsilon y u_0 \qquad \text{on body,} \quad 0 < y < \varepsilon/2 . \qquad (59b)$$

In terms of near-field variables, the last condition can be written:

$$\frac{\partial\tilde{\Phi}_1}{\partial X} = -u_0(1+2\varepsilon^4 Y) \qquad \text{for} \quad X = 0 , \quad -1/2\varepsilon^4 < Y < 0 . \qquad (60)$$

It would be consistent at this point to simplify this condition to $\partial\tilde{\Phi}_1/\partial X = -u_0$ and to apply it in $-\infty < Y < 0$. However, this would lead us to some undefined integrals, and so we use (60) as stated above. Consistency can be achieved later. We supplement (60) with the further condition:

$$\frac{\partial\tilde{\Phi}_1}{\partial x} = 0 \qquad \text{for} \quad X = 0 , \quad Y < -1/2\varepsilon^4 . \qquad (60')$$

This is consistent with (59a) in a small region in which $y = O(\varepsilon)$.

Finally, it should be noted that, to leading order, $\tilde{\Phi}_1$ satisfies the Laplace equation:

$$\tilde{\Phi}_{1XX} + \tilde{\Phi}_{1YY} = 0 \quad \text{in the fluid region.} \tag{61}$$

The problem just formulated for $\tilde{\Phi}_1$ would be straightforward to solve except for one difficulty: The free-surface condition, (58), is to be satisfied on $Y = \tilde{H}_1 + \ldots$. Usually in such problems it is easy to show that the boundary condition can be transferred to the undisturbed surface with negligible error, but such an operation is not trivial in the present problem. In terms of near-field variables, we must note that $\partial/\partial Y = O(1)$ and $\tilde{H}_1 = O(1)$, and so a simple Taylor expansion cannot be used. In terms of the original physical variables, we have $\partial\tilde{\Phi}_1/\partial y = O(\tilde{\Phi}_1/\varepsilon^5)$ and the boundary has to be moved a distance $\varepsilon H - \varepsilon\bar{H} = \varepsilon^5\tilde{H}_1 + \ldots = O(\varepsilon^5)$. This shows again that the transfer is not trivial.

Nevertheless, it can be carried out. Our demonstration is not rigorous, but it is convincing. We suppose first that (58) can be applied on $Y = 0$. Then we show that the solution so obtained satisfies the same condition applied on $Y = \tilde{H}_1$ (to an acceptable accuracy). We then presume that the reverse is true, that is, that a solution satisfying (58) (as stated) also satisfies (58) approximately when it is imposed on $Y = 0$.

If (58) is satisfied on $Y = 0$, we expect the solution to take the form

$$\tilde{\Phi}_1(X,Y) = F_0(x,y) + F_1(x,y)\exp\{iS(x,y)/\varepsilon^5\} , \tag{62}$$

where the functions $F_0(x,y)$, $F_1(x,y)$, and $S(x,y)$ all vary "slowly" with x and y , that is, $\partial/\partial x$ and $\partial/\partial y = O(1)$ when operating on these functions. (Such a result is well-known for the corresponding "wavemaker problem." See Appendix B.) In fact, in the near field, $F_1(x,y)$ is a constant, and

$$\exp\{iS(x,y)/\varepsilon^5\} = \exp\{i\nu(X + iY)\} ,$$

where

$$\nu = 1/u_0^2 ,$$

and so we see explicitly that $S(x,y) = \nu\{(x-x_0) + i[y-\varepsilon\bar{H}(x)]\}$, which indeed

varies slowly with x and y. The function $F_0(x,y)$ represents a nonwave-like motion, a local effect, that decays with distance from the body.

There is negligible difference whether we evaluate $F_0(x,y)$, $F_1(x,y)$, and $S(x,y)$ on $Y = 0$ or on $Y = \tilde{H}_1 + \ldots$, and so we need consider only the exponential factor in (62), which does change rapidly with x and y. We observe that

$$\begin{aligned}\exp\{iS(x,\varepsilon H)/\varepsilon^5\} &= \exp\{i[S(x,\varepsilon\bar{H}) + \varepsilon(H-\bar{H})S_y(x,\varepsilon\bar{H}) + \ldots]/\varepsilon^5\} \\ &= \exp\{iS(x,\varepsilon\bar{H})/\varepsilon^5\}\cdot\exp\{i\tilde{H}_1 S_y(x,\varepsilon\bar{H})\}\cdot\{1 + o(1)\} \quad .\end{aligned}$$

Thus, if we evaluate the exponential function on $Y = \tilde{H}_1 + \ldots$ (which is equivalent to $y = \varepsilon H$) instead of on $Y = 0$ (equivalent to $y = \varepsilon\bar{H}$), we effectively multiply by a factor $\exp\{i\tilde{H}_1 S_y(x,\varepsilon\bar{H})\}$.

Now we assume explicitly that (62) satisfies (58) on $Y = 0$, that is,

$$\begin{aligned}\left(\tilde{\Phi}_{1Y} + u_0^2\tilde{\Phi}_{1XX}\right)\Big|_{Y=0} &= \left(F_{0Y} + u_0^2 F_{0XX}\right)\Big|_{Y=0} \\ &\quad + \left(\frac{\partial}{\partial Y} + u_0^2\frac{\partial^2}{\partial X^2}\right)[F_1\exp\{iS/\varepsilon^5\}]\Big|_{Y=0} = 0 \ .\end{aligned}$$

The first term on the right-hand side varies slowly in x, whereas the second term varies very rapidly. Then the sum can equal zero only if each term separately equals zero. So we have

$$\left.\begin{aligned}F_{0Y} + u_0^2 F_{0XX} &= 0 \\ \left(\frac{\partial}{\partial Y} + u_0^2\frac{\partial^2}{\partial X^2}\right)[F_1\exp\{iS/\varepsilon^5\}] &= 0\end{aligned}\right\} \quad \text{on} \quad Y = 0 \quad (y = \varepsilon\bar{H}) \ .$$

On the other hand, let us evaluate the left-hand side of (58) on $Y = \tilde{H}_1 + \ldots$:

$$\begin{aligned}\left(\tilde{\Phi}_{1Y} + u_0^2\tilde{\Phi}_{1XX}\right)\Big|_{Y=\tilde{H}_1+\ldots} &= \left(F_{0Y} + u_0^2 F_{0XX}\right)\Big|_{Y=0} + \ldots \\ &\quad + \exp\{i\tilde{H}_1 S_y(x,\varepsilon\bar{H})\}\left(\frac{\partial}{\partial Y} + u_0^2\frac{\partial^2}{\partial X^2}\right)[F_1\exp(iS/\varepsilon^5)]\Big|_{Y=0} + \ldots \ .\end{aligned}$$

Each term on the right-hand side is separately equal to zero (to the order of magnitude considered here), and so we see that (58) is approximately valid if applied on $Y = \tilde{H}_1 + \ldots$. Thus we have shown the stated result: If (58) is satisfied on $Y = 0$ it is also satisfied approximately on $Y = \tilde{H}_1 + \ldots$. Now we assume that the converse is true, which means that we can simplify our boundary-value problem by imposing (58) on $Y = 0$.

The solution will be constructed with the following Green function:

$$G(X,Y;\xi,\eta) = \frac{1}{2\pi}\log\left[(X-\xi)^2+(Y-\eta)^2\right]^{1/2} + \frac{1}{2\pi}\log\left[(X-\xi)^2+(Y+\eta)^2\right]^{1/2}$$
$$+ \frac{1}{\pi}\int\!\!\!\!\!-_{0}^{\infty}\frac{dk}{k-\nu}\,e^{k(Y+\eta)}\cos k(X-\xi) - M\,e^{\nu(Y+\eta)}\sin\nu(X-\xi)\ . \qquad (63)$$

M is a constant to be determined. The integral is to be interpreted in a principal-value sense (denoted by the bar through the integral sign). This Green function gives the potential at (X,Y) corresponding to a unit source located at (ξ,η). However, an additional wave disturbance has been introduced with the M term. The above Green function satisfies the Laplace equation and the free-surface condition, (58). It represents the following wave motion very far away:

$$G_X(X,Y;\xi,\eta) \sim -\nu(M\pm 1)\,e^{\nu(Y+\eta)}\cos\nu(X-\xi) \quad \text{as} \quad X-\xi \to \pm\infty\ . \qquad (64)$$

Using this fundamental solution, we express the solution of our problem in the following form:

$$\tilde{\Phi}_1(X,Y) = \int_{-\infty}^{0}\mu(\eta)\,G(X,Y;0,\eta)\,d\eta + D\,e^{\nu Y}\cos\nu X\ , \qquad (65)$$

where $\mu(\eta)$ is an unknown function to be determined so that the body boundary condition, (60) and (60'), is satisfied. The extra term on the right-hand side of (65) is easily recognized as a solution of the homogeneous problem, since it yields no contribution to $\tilde{\Phi}_{1X}$ at $X = 0$.

We substitute (65) into the body boundary condition, obtaining:

$$\frac{\partial\tilde{\Phi}_1}{\partial X} = \frac{1}{2}\mu(Y) - \nu M e^{\nu Y}\int_{-\infty}^{0}\mu(\eta)e^{\nu\eta}d\eta = \begin{cases} -\,u_0(1+2\varepsilon^4 Y)\ , & -1/2\varepsilon^4 < Y < 0\ , \\ 0\ , & Y < -1/2\varepsilon^4\ . \end{cases} \qquad (66)$$

This is a very simple integral equation for $\mu(Y)$. The solution can obviously be expressed in the form:

$$\mu(Y) = \mu_0(Y) + E\,e^{\nu Y}\ , \qquad (67)$$

where

$$\mu_0(Y) = \begin{cases} -\,2u_0(1+2\varepsilon^4 Y)\ , & -1/2\varepsilon^4 < Y < 0\ , \\ 0\ , & Y < -1/2\varepsilon^4\ . \end{cases} \qquad (68)$$

Substituting (67) and (68) back into (66), we obtain the value of E:

$$E = -\frac{4u_0M}{1-M}\left(1 - \frac{2\varepsilon^4}{\nu}\left\{1 - e^{-\nu/2\varepsilon^4}\right\}\right) . \tag{69}$$

The constants M and D are not yet known, but otherwise we know everything in (65), and so we have obtained a solution that satisfies (i) the Laplace equation, (ii) the free-surface condition, and (iii) the body condition.

Earlier, we assumed that there was a stagnation point at $(x_0, \varepsilon/2)$ (or $X = 0$, $Y = 0$). We now find that this condition is sufficient for determining the constants M and D . From (54) and (53), we have

$$\frac{\partial\Phi}{\partial y} = \frac{\partial\bar{\Phi}}{\partial y} + \varepsilon^2\frac{\partial\tilde{\Phi}_1}{\partial Y} + \dots . \tag{70}$$

From the formulation of the $\bar{\Phi}$ problem, we know that

$$\frac{\partial\bar{\Phi}}{\partial y} \simeq -\varepsilon\,\bar{\Phi}_x^2\,\bar{\Phi}_{xx} = -\varepsilon^5C^2(0)\phi_{0_{xx}}^3(x_0,0) + o(\varepsilon^5) , \tag{71}$$

the last estimate being valid at the stagnation point (See Appendix A). So we must have

$$\frac{\partial\tilde{\Phi}_1}{\partial Y} = +\varepsilon^3C^2(0)\phi_{0_{xx}}^3(x_0,0) = \varepsilon^3u_0^2\phi_{0_{xx}}(x_0,0) \tag{72}$$

at $X = 0$, $Y = 0$ in order that $\partial\Phi/\partial y$, as expressed in (70), may vanish at the stagnation point. However, from the solution as given in (65), we find that $\partial\tilde{\Phi}_1/\partial Y$ is undefined (infinite) at this point unless $\mu(0) = 0$. To see this explicitly, we differentiate (65):

$$\left.\frac{\partial\tilde{\Phi}_1}{\partial Y}\right|_{X=0} = \int_{-\infty}^{0}\mu(\eta)\,\frac{\partial G(0,Y;0,\eta)}{\partial Y}\,d\eta + \nu D e^{\nu Y} . \tag{73}$$

From the definition of the Green function, (63), we have

$$\left.\frac{\partial G}{\partial Y}\right|_{X=\xi=0} = \frac{1}{2\pi}\left(\frac{1}{Y-\eta} + \frac{1}{Y+\eta}\right) + \frac{1}{\pi}\int_0^\infty\frac{dk\,k\,e^{k(Y+\eta)}}{k-\nu}$$
$$\rightarrow \frac{1}{\pi}\int_0^\infty\frac{dk\,k\,e^{k\eta}}{k-\nu} \quad \text{as} \quad Y \rightarrow 0 .$$

The last expression behaves as $-1/\pi\eta$ as $\eta \rightarrow 0$, which shows that the integral in (73) diverges as $Y \rightarrow 0$, unless it happens that $\mu(0) = 0$. So we now impose such a condition on μ , and this then determines M :

$$\begin{aligned} 0 = \mu(0) &= \mu_0(0) + E \\ &= -2u_0 - \frac{4u_0M}{1-M}\left(1 - \frac{2\varepsilon^4}{\nu}\left\{1 - e^{-\nu/2\varepsilon^4}\right\}\right) , \end{aligned}$$

where we have used (67), (68), and (69). Solving for M , we obtain:

$$M = -\frac{1}{1 - (4\varepsilon^4/\nu)[1 - \exp(-\nu/2\varepsilon^4)]} \approx -\frac{1}{1 - 4\varepsilon^4/\nu} \qquad (74)$$

It is useful also to note the following consequences:

$$E = -\mu_0(0) = 2u_0 , \qquad (75)$$

$$\mu(Y) = \begin{cases} 2u_0\{e^{\nu Y} - 1 - 2\varepsilon^4 Y\} , & -1/2\varepsilon^4 < Y < 0 , \\ 2u_0 e^{\nu Y} , & Y < -1/2\varepsilon^4 . \end{cases} \qquad (76)$$

The above choice of M guarantees that $\tilde{\Phi}_{1Y}$ remains finite at the stagnation point, but it does not yet lead to (72). In order to ensure that (72) is satisfied, we must evaluate the integral in (73) as $Y \to 0$ and then choose the value of D appropriately. The evaluation of the integrals will be found in Appendix C. From these results, we obtain:

$$D = \varepsilon^3 u_0^2 \phi_{0_{xx}}(x_0,0)/\nu = \varepsilon^3 u_0^4 \phi_{0_{xx}}(x_0,0) = \varepsilon^3 C^4(0)\phi^5_{0_{xx}}(x_0,0) . \qquad (77)$$

In summary, we note that the solution in the inner region is expanded as in (54), $\bar{\Phi}$ being the naive solution that was worked out in (I). The next term, $\varepsilon^7\tilde{\Phi}_1$, is obtained in the form given in (65), with $\mu(Y)$ given in (76) and D in (77). This completes the inner solution for our purposes.

Before matching this inner solution with the outer solution, it is worthwhile to comment on some unusual aspects of the foregoing analysis:

(i) Normally in using matched asymptotic expansions, one obtains inner and outer expansions each of which is nonunique in some way. Matching of the two expansions then removes the nonuniqueness in each. However, in our problem, we have obtained a complete (unique) near-field solution, at least to leading order of magnitude. This appears at first sight to eliminate the possibility of satisfying a radiation condition in the far field. In fact, this is precisely what happens, and fortunately too, for there is no radiation condition possible in the far field. In steady-motion problems, we usually specify that there should be no waves upstream; this is an adequate radiation condition in two dimensions. In our problem, however, the downstream waves appear only in a surface layer of vanishing thickness, and these waves are effectively isolated from the upstream fluid region. One might say that there is no upstream region at all, at least insofar as it might affect the waves

downstream of the body. Then one must seek an alternative condition to make the solution unique. As described above, we have used the solution behavior at the stagnation point to provide such a condition.

(ii) The Green function introduced in (63) is rather unusual because of the presence of the last term. The contribution of this term to $\tilde{\phi}_1$ is interesting. Effectively, it produces a second solution of the homogeneous problem. The one obvious solution of the homogeneous problem is the last term in (65). But one can construct others if some degree of singularity is tolerated at the origin. The M term in the Green function produces a solution of the homogeneous problem with precisely the singular character at the origin that is needed to cancel the singular velocity that would otherwise arise from the source distribution. This is what was accomplished in deriving the value of M in (74).

IV. MATCHING OF INNER AND OUTER SOLUTIONS

We now use the inner solution, which is completely known, to determine the unknown amplitude and phase of the waves in the outer region. From (8), (10), and (38), the outer solution can be written:

$$\Phi(x,y) = \bar{\Phi}(x,y;\varepsilon) + \varepsilon^{\alpha+1} \exp\left[(Y-\bar{H})/\bar{\Phi}_x^2\right] \mathcal{Re}\left\{A(x,y)\exp\left[i\Theta(x)\right]\right\} . \qquad (78)$$

The rapidly varying phase function, $\Theta(x)$, was given in (37); the complex amplitude function, $A(x,y)$, was partially determined:

$$A(x,0) = A_0/\bar{\Phi}_x(x,\varepsilon\bar{H}(x)) , \qquad (79)$$

where A_0 is a complex constant. All that remains to be determined is A_0 and α. We accomplish this by matching $\Phi_x(x,\varepsilon\bar{H})$ in the two regions.

For the moment, let us omit the $\mathcal{Re}$ notation in (78), just as we did earlier. We write out $\Phi_x(x,y)$:

$$\Phi_x(x,y) = \bar{\Phi}_x(x,y;\varepsilon) + \varepsilon^{\alpha+1}\left(A_x(x,y) + A(x,y)\left\{-\frac{\bar{H}'}{\bar{\Phi}_x^2} - \frac{2(Y-\bar{H})\bar{\Phi}_{xx}}{\bar{\Phi}_x^3} + i\Theta'(x)\right\}\right)$$
$$\cdot \exp\left[(Y-\bar{H})/\bar{\Phi}_x^2\right] \cdot \exp\left[i\Theta(x)\right] . \qquad (80)$$

On $y = \varepsilon\bar{H}$ (or $Y = \bar{H}$ *), this simplifies:

$$\Phi_x(x,\varepsilon\bar{H}) = \bar{\Phi}_x(x,\varepsilon\bar{H};\varepsilon) + \varepsilon^{\alpha+1}\left(A_x(x,0) + A(x,0)\left\{-\frac{\bar{H}'}{\bar{\Phi}_x^2} + i\Theta'(x)\right\}\right) \cdot \exp\left[i\Theta(x)\right]$$
$$+ \ldots . \qquad (80')$$

Now we change over to near-field variables, as defined in (52). We note that

$$\Theta'(x) = -\frac{1}{\varepsilon\bar{\Phi}_x^2(x,\varepsilon\bar{H})} \sim -\frac{1}{\varepsilon^5 u_0^2} \quad \text{as} \quad X \to 0 ;$$

$$\Theta(x) = -\frac{1}{\varepsilon}\int_{x_0}^{x} \frac{d\xi}{\varepsilon^4 u_0^2[1+O(\varepsilon^3)]} \sim -\frac{x-x_0}{\varepsilon^5 u_0^2} = -\frac{X}{u_0^2} .$$

---

*Note the difference in the definition of Y in the inner and outer regions: $Y = y/\varepsilon$ in the outer region (see (13)), whereas $Y = [y-\varepsilon\bar{H}(x)]/\varepsilon^5$ in the inner region (see (52)).

The estimate (48a) has been used here. Now we keep just the leading term in (80'), with X fixed:

$$\Phi_x(x,\varepsilon\bar{H}) \sim \bar{\Phi}_x(x,\varepsilon\bar{H};\varepsilon) - i(A_0/u_0^3)\,\varepsilon^{\alpha-6}\,e^{-iX/u_0^2} \quad . \tag{81}$$

This is the result that we will match to the inner solution.

The inner solution was given in (54), (65), (76), and (77). Let us find the derivative that matches (81):

$$\begin{aligned}\Phi_x(x,y) &= \bar{\Phi}_x(x,y;\varepsilon) + \varepsilon^7[\tilde{\Phi}_{1X}X_x + \tilde{\Phi}_{1Y}Y_x] + \ldots \\ &= \bar{\Phi}_x(x,y;\varepsilon) + \varepsilon^2\tilde{\Phi}_{1X} + \ldots \;.\end{aligned}$$

We must evaluate this as $X \to \infty$ in order to perform the matching. Introducing (64) into (65), differentiating with respect to X, and evaluating the resulting integral, we find that

$$\tilde{\Phi}_{1X} \sim -\nu\left\{4\varepsilon^4 u_0 \cos \nu X + D \sin \nu X\right\} e^{\nu Y} \sim -\varepsilon^5 C^2(0)\phi_{0xx}^3(x_0,0)\sin X/u_0^2 + \ldots \;. \tag{82}$$

In (82), the term containing $\cos \nu X$ comes from the integral in (65), and the term containing $\sin \nu X$ comes from the homogeneous solution in (65). As is apparent here, the latter is of lower order of magnitude and thus dominates in (82).

Now we match these near-field results with the real part of (81), the result being that

$$A_0 = -C^5(0)\phi_{0xx}^6(x_0,0) \quad . \tag{83}$$

We also find that

$$\alpha = 11 \quad . \tag{84}$$

We can substitute back into the formulas for Φ or for Φ_x in the far field. The latter, in particular, gives us:

$$\Phi_x(x,\varepsilon\bar{H}(x)) = \bar{\Phi}_x(x,\varepsilon\bar{H};\varepsilon) + i\varepsilon^{11}\frac{C^5(0)\phi_{0xx}^6(x_0,0)}{\bar{\Phi}_x^3}\exp\left(-\frac{i}{\varepsilon}\int_{x_0}^{x}\frac{d\xi}{\bar{\Phi}_x^2}\right) + \ldots \;. \tag{85}$$

This is our final form of the solution in terms of Φ .

V. WAVE RESISTANCE

Exact nonlinear formulas are readily available for computing wave resistance if the fluid velocity and free-surface disturbance are known. From Wehausen and Laitone (1960), Equation (8.6), for example, we have:

$$R = \frac{1}{2}\rho U^2 L \int_{-\infty}^{\varepsilon H(x)} dy \left\{\phi_y^2(x,y) - [\phi_x(x,y) - 1]^2\right\} + \frac{1}{2}\rho g L^2 \varepsilon^2 H^2(x) \ , \tag{86}$$

where R is the wave resistance, and all other quantities are as previously defined. The right-hand side can be computed at any x downstream of the body, but we simplify the task by letting $x \to \infty$.

The general far-field expression for $\phi_x(x,y)$ has been given in (80). As $x \to +\infty$, the following approximations can all be used:

$$\left.\begin{aligned} &\bar{\Phi}_x \simeq 1 \ , \quad \bar{\Phi}_{xx} \simeq 0 \ , \quad \bar{\Phi}_y \simeq 0 \ ; \\ &A(x,y) \simeq A_0 \ , \quad A_x \simeq 0 \ ; \\ &\bar{H} \simeq 0 \ , \quad \bar{H}' \simeq 0 \ ; \\ &\Theta'(x) \simeq -1/\varepsilon \ . \end{aligned}\right\} \tag{87}$$

Noting that A_0 is a real constant and taking the appropriate real parts in (80) (see (78)), we obtain the asymptotic estimate,

$$\phi_x(x,y) \simeq 1 + \varepsilon^{11} A_0 e^{y/\varepsilon} \sin[\Theta(x)] \ , \tag{88a}$$

valid as $x \to +\infty$. Similarly we find that

$$\phi_y(x,y) \simeq \varepsilon^{11} A_0 e^{y/\varepsilon} \cos[\Theta(x)] \ . \tag{88b}$$

Moreover, from (9), (11), (39), and (40), we also have:

$$H(x) \simeq -\varepsilon^{11} A_0 \sin[\Theta(x)] \ . \tag{88c}$$

Carrying out the integration in (86), with the upper limit of the integral replaced (consistently) by zero, we obtain finally that

$$C_W = \frac{R}{(1/2)\rho U^2 L} \simeq \frac{1}{2}\varepsilon^{23} A_0^2 = \frac{1}{2}\varepsilon^{23} C^{10}(0)\phi_{0_{xx}}^{12}(x_0,0) \ , \tag{89}$$

where, as before, $C(0)$ is the body curvature at the intersection of the body and the undisturbed free surface and $\phi_0(x,y)$ is the first term in the naive expansion (the potential for the double-body flow). C_W is the wave-resistance coefficient for the 2-D body.

It can hardly be surprising that the wave resistance depends on the shape of the body only near the free surface, since the wave motion occurs entirely in a very thin layer, in fact, in a vanishingly thin layer, near the level of the free surface, and the wave motion is induced by the peculiar nature of the streaming flow on the body near this level (see (I)). Furthermore, we assumed that the body has a vertical tangent here, and so it is also reasonable to expect that the curvature should have a dominant effect in creating waves.

If the curvature were zero at $y = 0$, we would expect the generation of waves to depend primarily on higher-order derivatives of the body shape in this same region. Presumably wave amplitude and wave resistance would then be of even higher order than in the case presented.

We can only speculate now what would be the result if the body contour were analytically straight in some finite neighborhood of $y = 0$. Our speculation is that waves and wave resistance would be small of exponential order as in the case of a submerged body.

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APPENDIX A. STAGNATION-POINT CONDITIONS ON $\bar{\Phi}$

Here we obtain the estimate given in (48c) and (71).

From (I), from the definition of the naive expansion,

$$\bar{\Phi}_x \sim \phi_{0_x} + \varepsilon\phi_{1_x} + \dots \quad .$$

On the body very close to the free surface,

$$\phi_{j_x} \simeq - \phi_{j_n} \quad .$$

If the body shape has continuous curvature and a vertical tangent at $y = 0$, we have, from Equation (42) of (I),

$$\phi_{0_x} = O(\varepsilon^3) \quad .$$

From Equation (62) of (I),

$$\phi_{1_x}\Big|_{\text{body}} \simeq - 2\,y\,C(0)\phi_{0_{xx}}(x_0,0) \qquad \text{for} \quad 0 < y < \varepsilon/2 \ ,$$

where $C(0)$ and $\phi_0(x,y)$ are given by (48d) and (48e). Putting these results together gives us:

$$\bar{\Phi}_x(x_0,\tfrac{\varepsilon}{2}) \simeq - \varepsilon^2 C(0)\phi_{0_{xx}}(x_0,0) \quad , \tag{A.1}$$

the relationship given in (48a).

To obtain (71) (and also (48b)), we start from the free-surface condition (5), which is still valid (by definition) if Φ is replaced by $\bar{\Phi}$. Thus

$$\bar{\Phi}_y = -\varepsilon^2\{\bar{\Phi}_x^2\bar{\Phi}_{xx} + 2\bar{\Phi}_x\bar{\Phi}_y\bar{\Phi}_{xy} + \bar{\Phi}_y^2\bar{\Phi}_{yy}\} \quad \text{on} \quad y = \varepsilon\bar{H}(x) \quad .$$

On the right-hand side, the first term is the lowest-order term, and so, substituting from (A.1), we obtain:

$$\bar{\Phi}_y(x_0,\tfrac{\varepsilon}{2}) \simeq - \varepsilon^5 C^2(0)\phi^3_{0_{xx}}(x_0,0) \ ,$$

the desired result, (71).

APPENDIX B. COMPONENTS OF THE INNER SOLUTION

In (62), the inner solution was written as the sum of two parts:

$$\tilde{\Phi}_i(X,Y) = F_0(x,y) + F_1(x,y) \exp\{iS(x,y)/\varepsilon^5\} \quad , \tag{62}$$

where it was stated that $F_0(x,y)$, $F_1(x,y)$, and $S(x,y)$ all vary "slowly" with x and y , although this expression represents waves of very small wavelength.

Such statements can be verified directly in the analogous "wavemaker problem:"

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{in} \quad \begin{cases} x > 0 \;, \\ y < 0 \;, \end{cases} \tag{B.1}$$

$$\phi_x = f(y) \quad \text{on} \quad x = 0 \;, \quad y < 0 \;, \tag{B.2}$$

$$\nu\phi - \phi_y = 0 \quad \text{on} \quad y = 0 \;, \quad x > 0 \tag{B.3}$$

This is the mathematical statement of the problem if the vertical wall, $x = 0$, moves with normal velocity component $f(y)\exp(i\omega t)$, where ω is the radian frequency of the motion and $\nu = \omega^2/g$ is the corresponding wave number. The potential for the problem is $\phi(x,y)\exp(i\omega t)$. Of course, a radiation condition must also be imposed. This problem is very similar to the near-field problem considered in Section III.

The wavemaker problem has been studied by many people, and we can write down the solution directly:

$$\phi(x,y) = \tilde{\phi}(x,y) + \hat{\phi}(x,y) \;, \tag{B.4}$$

where

$$\tilde{\phi}(x,y) = e^{\nu y} \, [A_0 \sin \nu x + B \cos \nu x] \;, \tag{B.5}$$

$$\hat{\phi}(x,y) = \int_0^\infty dk \; A(k) e^{-kx} (k \cos ky + \nu \sin ky) \;. \tag{B.6}$$

The term in (B.5) containing B is the solution of the homogeneous problem, and so B is arbitrary. The constant A_0 in (B.5) and the function $A(k)$

in (B.6) are given by the following:

$$A_0 \;=\; 2\int_{-\infty}^{0} dy\, f(y)\, e^{\nu y} \quad ; \tag{B.7}$$

$$A(k) \;=\; -\frac{2}{\pi k(k^2+\nu^2)}\int_{-\infty}^{0} dy\, f(y)\,(k\cos ky + \nu \sin ky) \quad . \tag{B.8}$$

In the above problem, if we allow ν (or ω) to become very large, it is immediately obvious that $\tilde{\phi}_x$ and $\tilde{\phi}_y$ are $O(\nu\tilde{\phi})$. We can express this symbolically by writing

$$\frac{\partial}{\partial x}\,,\;\frac{\partial}{\partial y} \;=\; O(\nu) \quad . \tag{B.9}$$

Since $\tilde{\phi}$ represents waves of wavenumber ν , this is all rather obvious.

What is not so obvious is that

$$\frac{\partial}{\partial x}\,,\;\frac{\partial}{\partial y} \;=\; O(1) \tag{B.10}$$

when they operate on $\hat{\phi}(x,y)$, even if $\nu \to \infty$. To show this, substitute (B.8) into (B.6), interchange order of integration, and write the potential in terms of a function of a complex variable $z = x + iy$:

$$\begin{aligned}\hat{\phi}(x,y) \;&=\; -\frac{2}{\pi}\,\mathit{Re}\left(\int_{-\infty}^{0} d\eta\, f(\eta)\int_{0}^{\infty}\frac{dk\;e^{-kz}}{k(k-i\nu)}\,(k\cos k\eta + \nu\sin k\eta)\right)\\ &=\; \frac{1}{\pi}\,\mathit{Re}\left(\int_{-\infty}^{0} d\eta\, f(\eta)\left(\log\left\{\frac{z-i\eta}{z+i\eta}\right\} - 2e^{-i\nu(z+i\eta)}E_1(-i\nu(z+i\eta))\right)\right) \quad ,\end{aligned}$$

where

$$E_1(u) \;=\; \int_{u}^{\infty}\frac{dt\;e^{-t}}{t} \quad , \text{ the exponential integral} \quad .$$

As $\nu \to \infty$, note that

$$E_1(\nu u) \;\sim\; \frac{e^{-\nu u}}{\nu u} \qquad \text{for} \quad |\arg u| < 3\pi/2 \quad .$$

Using this asymptotic estimate, we now find that, as $\nu \to \infty$,

$$\hat{\phi}_x - i\hat{\phi}_y \;=\; \frac{1}{\pi}\int_{-\infty}^{0} d\eta\, f(\eta)\left(\frac{1}{z-i\eta} - \frac{1}{z+i\eta} + O(1/\nu)\right) \quad ;$$

$$\hat{\phi}_{xx} - i\hat{\phi}_{xy} = -\frac{1}{\pi}\int_{-\infty}^{0} d\eta\, f(\eta)\left[\frac{1}{(z-i\eta)^2} - \frac{1}{(z+i\eta)^2} + O(1/\nu)\right] ;$$

etc. This is the result indicated by (B.10). It means that $\phi(x,y)$ varies "slowly" even if the wavenumber becomes asymptotically large.

In the problem in Section III, we had the free-surface condition

$$\nu\phi_y + \phi_{xx} = 0 \quad \text{on} \quad y = 0 , \tag{B.11}$$

instead of (B.3). The solution can be written as in (B.4), with $\tilde{\phi}(x,y)$ given by (B.5) and

$$\hat{\phi}(x,y) = \int_0^{\infty} dk\, A(k)\, e^{-kx}(\nu \cos ky - k \sin ky) . \tag{B.12}$$

In this problem, one can show that

$$A_0 = 2\int_{-\infty}^{0} dy\, f(y)\left\{1 - e^{\nu y}\right\} ; \tag{B.13}$$

$$A(k) = \frac{2}{\pi k(k^2+\nu^2)}\int_{-\infty}^{0} dy\, f(y)\left\{\nu(1 - \cos ky) + k \sin ky\right\} . \tag{B.14}$$

Then it is straightforward to show results identical to those obtained above for the wavemaker problem, as summarized in (B.9) and (B.10).

APPENDIX C. EVALUATION OF SEVERAL INTEGRALS

We chose the arbitrary constant M in the Green function (see (63)) to ensure that $\partial\tilde{\Phi}_1/\partial Y$ would be bounded at $X = 0$ even as $Y \to 0$. The result was given in (74). This left the constant D, which first appeared in (65), as a still undetermined quantity. Its value was to be found by substituting from (73) into (72). To carry this out, we have to evaluate the integral in (73) for $X = Y = 0$. Its value turns out to be negligible, but this result is not obvious, and so we derive it here.

From (73) and (72), with $Y = 0$ in the former, we have

$$\varepsilon^3 u_0^2 \phi_{0_{xx}}(x_0,0) - \nu D = \int_{-\infty}^{0} d\eta\, \mu(\xi) \left.\frac{\partial G}{\partial Y}\right|_{\substack{X=Y=0\\ \xi=0}} . \tag{C.1}$$

(Recall that $\nu = 1/u_0^2$.) Following (73), it was shown that

$$\left.\frac{\partial G}{\partial Y}\right|_{\substack{X=Y=0\\ \xi=0}} = \frac{1}{\pi} \int_0^{\infty} \frac{dk\, k\, e^{k\eta}}{k - \nu} .$$

The value of $\mu(\eta)$ was given in (76). We substitute these results into (C.1):

$$\varepsilon^3 u_0^2 \phi_{0_{xx}}(x_0,0) - \nu D = \frac{2u_0}{\pi} \int_{-\infty}^{0} d\eta\; e^{\nu\eta} \int_0^{\infty} \frac{dk\, k\, e^{k\eta}}{k-\nu}$$

$$- \frac{2u_0}{\pi} \int_{-1/2\varepsilon^4}^{0} d\eta\; (1 + 2\varepsilon^4\eta) \int_0^{\infty} \frac{dk\, k\, e^{k\eta}}{k-\nu}$$

$$= \frac{4u_0\varepsilon^4}{\pi\nu} \int_0^{\infty} dk \left\{\frac{1}{k-\nu} - \frac{1}{k}\right\}\left\{1 - e^{-k/2\varepsilon^4}\right\} \tag{C.2}$$

$$\sim -\frac{4u_0\varepsilon^4}{\pi\nu} \left\{\gamma + \log\frac{\nu}{2\varepsilon^4} - \frac{2\varepsilon^4}{\nu} + \ldots\right\} \quad \text{as} \quad \nu \to \infty , \tag{C.3}$$

where γ is Euler's constant. The expression in (C.2) is found after some manipulation of the preceding line, and the estimate in (C.3) follows from standard asymptotic estimates of exponential integrals.

The right-hand side of (C.3) is $O(\varepsilon^4 \log {}^4)$, whereas the first term on the left-hand side is $O(\varepsilon^3)$. So we conclude that

$$D = \varepsilon^3 u_0^2 \phi_{0_{xx}}(x_0,0)/\nu , \tag{C.4}$$

as stated in (77).

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